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Study of the Phase-Lock Phenomenon for a Circular Slingatron

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The phase-lock phenomenon of a mass sled sliding along in a circular slingatron is studied both numerically and analytically. Parameters that describe a slingatron, in which the phase angle of the swing arms increases quadratically in time, are found to be simply related to the sled's speed during phase lock. The time required for phase lock to occur is related to a simple exponential function of the gyration speed and the coefficient of friction between the sled and track. Accurate time histories describing the motion of the accelerating sled are expressed in terms of confluent hypergeometric functions $_1F_1$. These results are then used to obtain physical insight into why the phase-lock phenomenon takes place and to describe the important role that friction plays by damping the oscillatory motion of the sled.

Nomenclature

a = complex parameter for the confluent hypergeometric function $_1F_1$
b = constant of integration
$C1, C2$ = complex constants of integration
$\boldsymbol{F}_{//}$ = force vector parallel to slingatron track $|\boldsymbol{F}_{//}| = F_{//}$
$\boldsymbol{F}_{\perp}$ = force vector perpendicular to slingatron track $|\boldsymbol{F}_{\perp}| = F_{\perp}$
$_1F_1$ = confluent hypergeometric function
f = frequency of gyration arm 1/s
$\dot{f}$ = angular acceleration of gyration arm 1/s^2
i = $\sqrt{-1}$
$\boldsymbol{i}$ = unit vector along ordinate
$\boldsymbol{j}$ = unit vector along abscissa
$\boldsymbol{k}$ = unit vector $\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$
M = mass of slingatron sled
O = order of magnitude
P = frequency ratio $= \dot{f}/f^2$
q = transform of independent variable
$\boldsymbol{R}$ = radius vector of slingatron circle $|\boldsymbol{R}| = R$
$\boldsymbol{r}$ = radius vector of gyration arm $|\boldsymbol{r}| = r$
t = time
$\boldsymbol{V}$ = velocity vector $V = \sqrt{(\dot{x}^2 + \dot{y}^2)}$
$\hat{\boldsymbol{V}}$ = velocity of sled relative to the track
x = ordinate of sled
y = abscissa of sled
z = transformed dependent variable
α = ratio of radii r/R
γ = polar angle
μ = coefficient of friction
ρ = transformed independent variable
ϕ = polar angle of vector $\boldsymbol{R}$
ψ = polar angle of vector $\boldsymbol{r}$
θ = difference of angles $= \psi - \phi$
$||$ = absolute value

Superscripts

$\cdot$ = d/dt
$'$ = d/dq, d/dρ, d/dϕ
$*$ = complex conjugate

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Introduction

A CONCEPT, called the *slingatron*, used to accelerate a mass (sled) to high velocity has been proposed and studied by D. A. Tidman.[1] This accelerator can propel either a constant mass or an ablating mass sled along a guide tube forming the slingatron track,[2] and several slingatron configurations have been examined.[1,2] A simple friction model is also used in which the friction force is assumed to be proportional to the normal force exerted by the track[2] on the moving sled.

Previous work has shown that the relative phase angle (between the sleds' position vector and the gyration vector of the guide tube) locks into an approximate constant value. This paper presents numerical and analytical results found from our investigation of this phase-lock phenomenon for a constant point mass accelerated around a circular track with the friction coefficient taken to be a constant. Numerical simulations, as well as simple analytical results, show that the mass can reach very high velocities, provided that the gyration speed can increase with sufficient acceleration. We will show, when the gyration phase angle varies quadratically with time, that the phase-lock value plus the time required to obtain this value are simply related to the gyration acceleration, the coefficient of friction, and the ratio of the two characteristic lengths that describes the slingatron geometry. These relationships are used to aid our understanding and to make predictions of the phase-lock phenomenon.

Equations of Motion

Consider the circular slingatron shown in Fig. 1, which has radius vector $\boldsymbol{R}$ at the sled location with its center attached to the gyration arm $\boldsymbol{r}$ positioned at angle ψ (t) with respect to the laboratory frame at time t. The position of the accelerating point mass is given by $\boldsymbol{R} + \boldsymbol{r}$, and the corresponding velocity is therefore $\boldsymbol{V} = \dot{\boldsymbol{R}} + \dot{\boldsymbol{r}}$. The force $\boldsymbol{F}$ acting on the mass M can be written as $\boldsymbol{F} = F_{\perp}\boldsymbol{m} + F_{//}\boldsymbol{n}$ in which $\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{k}$ and $\boldsymbol{n} = -\boldsymbol{R}'/R'$ are unit vectors pointing normal (toward the center of the circle) and antiparallel to the circular track. Letting $\boldsymbol{i}, \boldsymbol{j}$ be unit vectors along the laboratory frame's abscissa and ordinate gives the time derivative of the momentum equations the following form:

$$M\ddot{x} = -F_{\perp}\cos\phi + F_{//}\sin\phi$$
$$M\ddot{y} = -F_{\perp}\sin\phi - F_{//}\cos\phi \tag{1}$$

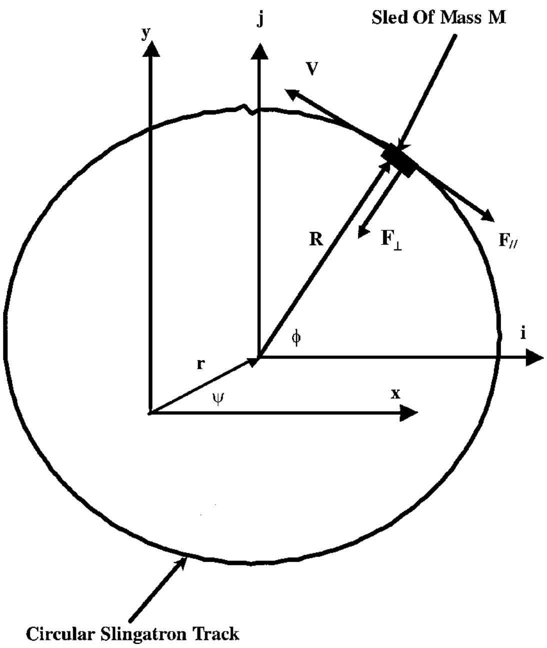

Fig. 1 Schematic of a circular slingatron.

with Cartesian coordinates (x, y) in the $(\boldsymbol{i},\boldsymbol{j})$ laboratory frame constrained so that

$$x = R\cos\phi + r\cos\psi, \qquad y = R\sin\phi + r\sin\psi \tag{2}$$

The parallel force $F_{//}$ attributed to friction is represented as

$$F_{//} = \mu F_{\perp} \tag{3}$$

Equations (1–3) will determine the angle ϕ in terms of the given angle $\psi(t)$ resulting from the following differential equation:

$$-\ddot{\phi} = \mu\dot{\phi}^2 + \alpha(\ddot{\psi} + \mu\dot{\psi}^2)\cos(\psi - \phi)$$
$$+\alpha(\mu\ddot{\psi} - \dot{\psi}^2)\sin(\psi - \phi) \tag{4}$$

for which the length ratio is defined as $\alpha = r/R$.

Generally the solution to Eq. (4) requires numerical integration for an arbitrary choice of the time-dependent gyration angle $\psi(t)$. The mass velocity $V^2 = \dot{x}^2 + \dot{y}^2$ relative to the laboratory reference frame is calculated easily using

$$V^2 = [2\alpha\cos(\psi - \phi)(\dot{\phi}/\dot{\psi}) + \alpha^2 + (\dot{\phi}/\dot{\psi})^2]\dot{\psi}^2R^2 \tag{5}$$

Phase Lock-In

D. A. Tidman[1] has presented a complete analysis of cases where phase lock-in occurs for all times t, that is, when the angle $\theta = \psi - \phi$ is constant so that $\dot{\phi} = \dot{\psi}$ and $\ddot{\phi} = \ddot{\psi}$. These conditions transform Eq. (4) to

$$\ddot{\phi} = b\dot{\phi}^2, \qquad b = \frac{\alpha(\sin\theta - \mu\cos\theta) - \mu}{\alpha(\mu\sin\theta + \cos\theta) + 1} \tag{6}$$

and this is easily integrated so that Eq. (5) becomes

$$V^2 = \frac{(\dot{\phi}_0 R)^2(2\alpha\cos\theta + \alpha^2 + 1)}{(b\dot{\phi}_0 t - 1)^2}$$

with

$$\phi_0 = \phi(0) \quad \text{and} \quad \dot{\phi}_0 = \dot{\phi}(0) \tag{7}$$

Examining the factor b, defined in the second equation of Eqs. (6), shows that the velocity is time independent when $b = 0$ or when the parameters are related as

$$\theta = \pm 2\tan^{-1}\left[\frac{\sqrt{(\alpha^2 - 1)\mu^2 + \alpha^2} \mp \alpha}{(\alpha - 1)\mu}\right] + 2n\pi,$$
$$n = \pm 0, 1, 2\cdots$$

so that

$$V^2 = \frac{\left[1 \pm \sqrt{(\alpha^2 - 1)\mu^2 + \alpha^2}\right]^2(\dot{\phi}_0 R)^2}{\mu^2 + 1} \tag{8}$$

A short amount of algebra reveals that Eq. (8) is satisfied whenever the force caused by friction $\mu F_{\perp}$ is large enough to prevent the point mass from accelerating in a direction tangent to the path, which this mass traverses. Furthermore, the velocity given by Eq. (7) was shown by Tidman[1] to diverge (to a nonrelativistic limit) at the time $t_{\infty} = (\dot{\phi}_0 b)^{-1}$. In practice, for this to occur one must have $t_{\infty} \geq 0$, which means that a divergent velocity V will occur under the assumption $\alpha\sqrt{(1 + \mu^2)} \leq 1$ whenever the parameters also satisfy the relation

$$\sin(\theta - \tan^{-1}\mu) \geq \mu/\alpha\sqrt{\mu^2 + 1} \tag{9}$$

An example typical of this behavior is shown in Fig. 2.

Driven Circular Slingatron

We next consider a class of slingatrons with the gyration of the circular track prescribed by the angular displacement $\psi(t) = \psi_0 + 2\pi f t + \pi\dot{f}t^2$ for constants f and $\dot{f}$. Typical numerical solutions to Eqs. (4) and (5) are given in Fig. 3 with initial conditions $\phi_0 = 0$ and $\dot{\phi}_0 = 2\pi f$. Two corresponding calculations of the angle $\theta = \psi - \phi$ as functions of time are shown in Fig. 4. The last two plots show the slingatron will eventually achieve phase lock-in (that is, $\dot{\theta} = 0$ and $\ddot{\theta} = 0$), when the gyration angle is quadratic in time, that is, $\psi(t) = \psi_0 + 2\pi f t + \pi\dot{f}t^2$.

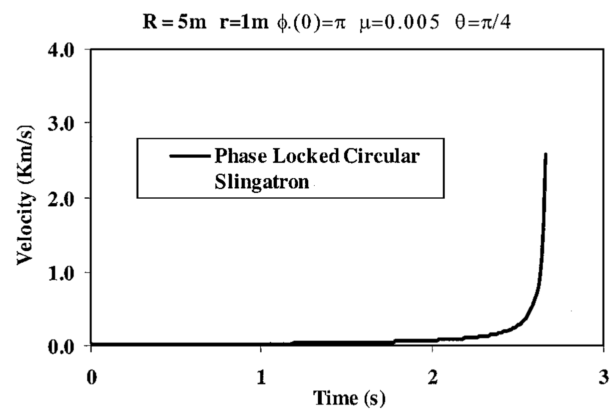

Fig. 2 Divergent velocity for R = 5 m, t_{∞} = 2.68 s.

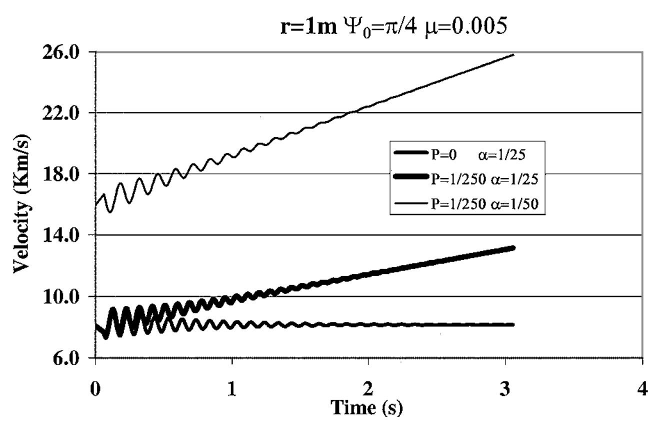

Fig. 3 Velocity vs time $P = \dot{f}/f^2$, $\alpha = r/R$.

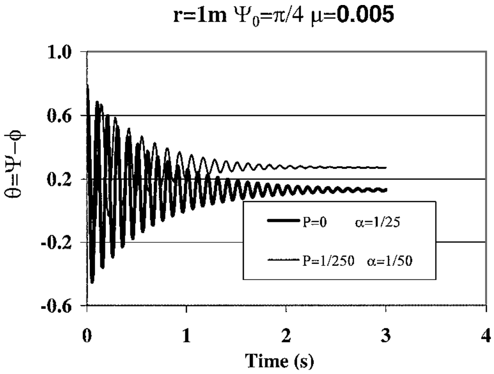

Fig. 4 Angle lock-in vs time $P = \dot{f}/f^2$ $\alpha = r/R$.

To understand better how the lock-in angle is related to the given parameters, we now rewrite Eq. (4) in terms of $\theta = \psi - \phi$, and calling $q = \dot{f}\,t + f$ gives us

$$\dot{f}^2\theta'' - \dot{f}^2\mu\theta'^2 + 4\pi\dot{f}\mu q\theta' + 2\pi(2\pi\lambda q^2 + \dot{f}\mu)\sin\theta - 2\pi(2\pi\mu q^2 - \dot{f}\lambda)\cos\theta = 2\pi(\pi\mu q^2 + \dot{f})$$

$$\lambda = \sqrt{(\alpha^2 - 1)\mu^2 + \alpha^2} \tag{10}$$

Focusing our attention on large values of t, consider the following asymptotic expansion for θ:

$$\theta = \theta_0 + \theta_2/q^2 + \theta_4/q^4 + \theta_6/q^6 + \cdots$$

Substituting this back into Eq. (10) and keeping only the largest order term gives us

$$[\alpha\sin(\theta_0) - \alpha\mu\cos(\theta_0) - \mu] = 0$$

$$\Rightarrow \lim_{t\to\infty}\frac{\theta_0}{2} \to \tan^{-1}\left[\frac{-\alpha \pm \sqrt{\alpha^2 + \mu^2(\alpha^2 - 1)}}{\mu(\alpha^2 - 1)}\right] + n\pi, \qquad n = 0, 1, 2\cdots$$

However, D. A. Tidman[1] has shown that θ values near to $\pi/2$ lead to instabilities, which tells us that the correct asymptotic must have the following form:

$$\lim_{t\to\infty}\frac{\theta_0}{2} \to 2n\pi + \tan^{-1}\left[\frac{-\alpha + \sqrt{\alpha^2 + \mu^2(\alpha^2 - 1)}}{\mu(\alpha^2 - 1)}\right] \approx \frac{(\alpha + 1)\mu}{\alpha} + 0(\mu^3), \qquad n = 0, 1\cdots \tag{11}$$

The limiting value plotted in Fig. 4 corresponds to Eq. (11) with $n = 0$. Subtracting this asymptotic value of θ so that $\gamma \to \theta + \theta_0$ transforms Eq. (10), after some algebra, to the following expression:

$$\dot{f}^2\gamma'' - \mu\dot{f}^2\gamma'^2 + 4\pi\mu\dot{f}q\gamma' + 2\pi(2\pi\lambda q^2 + \dot{f}\mu)\sin\gamma + 2\pi(2\pi\mu q^2 - \dot{f}\lambda)\cos\gamma = 2\pi(\mu\pi q^2 + \dot{f})$$

$$\lambda = \sqrt{(\alpha^2 - 1)\mu^2 + \alpha^2}, \qquad q = \dot{f}t + f \tag{12}$$

Because $\theta_0 \approx \mu + \mu/\alpha$ is small for realistic values of μ and α, we now make the small angle approximation $\gamma \ll 1$, reducing Eq. (12) to

$$\dot{f}^2\gamma'' + 4\pi\dot{f}\mu q\gamma' + 2\pi(2\pi\lambda q^2 + \dot{f}\mu)\gamma = 2\pi\dot{f}(\lambda + 1)$$

$$\gamma(f) = \psi_0 - \theta_0, \qquad \dot{\gamma}(f) = 0 \tag{13}$$

There are many ways to write solutions to Eq. (13), but we use the following procedure, which seemed expedient for both numerical and analytic examinations. First, make the substitutions

$$\rho = 2\Gamma q^2/\dot{f}, \qquad z = \gamma\exp[(\pi\mu + \Gamma)q^2/\dot{f}] \tag{14}$$

into the left-hand side of Eq. (13) giving us

$$\rho z'' + \left(\frac{1}{2} - \rho\right)z' + \frac{\rho(\pi^2\lambda^2 + \Gamma - \pi^2\mu) - \Gamma^2}{4\Gamma^2}z = 0$$

Next choose $\Gamma = \pm i\pi\sqrt{(\lambda - \mu^2)}$ because our interests imply $\lambda > \mu^2$ to force the coefficient of z to be independent of ρ. The left side of Eq. (13) is now transformed into the complex plane such that

$$\rho z'' + \left(\tfrac{1}{2} - \rho\right)z' - z/4 = 0 \tag{15}$$

The solution to Eq. (15) is a linear combination of the confluent hypergeometric function[3] ${}_1F_1$ so that the solution of the left-hand side of Eq. (13) γ_H is

$$\gamma_H(\rho) = \left[C1\,{}_1F_1\left(\tfrac{1}{4}, \tfrac{1}{2}, \rho\right) + C2\sqrt{\rho}\,{}_1F_1\left(\tfrac{3}{4}, \tfrac{3}{2}, \rho\right)\right] \times \exp\left[\left(\pm i\sqrt{\lambda - \mu^2} - \mu\right)\pi q^2/\dot{f}\right], \qquad \lambda - \mu^2 > 0 \tag{16}$$

for complex constants $C1$, $C2$. Equation (16) shows that γ_H form conjugate pairs so the required real values of γ are found by taking linear combinations (average values) of the two conjugates.

These results show that the amplitude of γ and therefore θ decreases as a linear combination of functions having factors

$$\begin{Bmatrix}\dot{f}t + f\\ 1\end{Bmatrix}\exp[-\pi\mu(\dot{f}t + f)^2/\dot{f}] \tag{17}$$

provided that the functions ${}_1F_1$ for the parameters used here do not diverge. Using the integral representation of ${}_1F_1$ shows

$${}_1F_1\left(\frac{1}{4}, \frac{1}{2}, \rho\right) = \frac{\sqrt{\pi}}{\Gamma(1/4)\Gamma(1/4)}\int_0^1 t^{-\frac{3}{4}}e^{\rho t}(1 - t)^{-\frac{3}{4}}\,\mathrm{d}t \tag{18}$$

Now take the absolute value of each side of Eq. (18), remembering that ρ is a pure imaginary; thus, we find

$$\left|{}_1F_1\left(\frac{1}{4}, \frac{1}{2}, \rho\right)\right| = \left|\frac{\sqrt{\pi}}{\Gamma(1/4)\Gamma(1/4)}\right|\left|\int_0^1 t^{-\frac{3}{4}}e^{\rho t}(1 - t)^{-\frac{3}{4}}\,\mathrm{d}t\right| \le \left|\frac{\sqrt{\pi}}{\Gamma(1/4)^2}\right|\int_0^1 t^{-\frac{3}{4}}(1 - t)^{-\frac{3}{4}}\,\mathrm{d}t = 1 \tag{19}$$

showing that ${}_1F_1\left(\frac{1}{4}, \frac{1}{2}, \rho\right)$ is bound and a similar argument shows that $|{}_1F_1\left(\frac{1}{4}, \frac{3}{2}, \rho\right)|$ is also bound. Thus, the homogeneous solutions, given by Eq. (16), decay according to Eq. (17). Obtaining solutions for the limiting case, in which $\dot{f} \to 0$, is easily performed directly from the first of Eq. (13) using $(') = 1/\dot{f}(\cdot)$ followed by $\dot{f} \to 0$. For these cases the solution θ has the form

$$\theta = \left[C1\cos\left(2\pi f\sqrt{\alpha - \mu^2}t\right) + C2\sin\left(2\pi f\sqrt{\alpha - \mu^2}t\right)\right] \times \exp(-2\pi f\mu t) + \mu(\alpha + 1)/\alpha \tag{20}$$

which shows that the decaying factors of Eqs. (17) are replaced with $\exp(-2\pi f\mu t)$.

The nonhomogeneous solution of Eq. (13) can be shown to asymptotically approach zero for large q. Therefore, the asymptotic solution found in Eq. (11) continues to show $\theta \to \theta_0 \approx \mu + \mu/\alpha$ for small values of μ and α. A sample from Fig. 4 that exhibits both the asymptotic and decaying characteristics of a solution to Eq. (13) is given in Fig. 5. The velocity corresponding to the asymptotic solution of Eq. (13) is obtained from Eq. (5) by letting $\dot{\phi}/\dot{\psi} \to 1$. Thus one will find

$$V \to \frac{\sqrt{(\alpha^2 - 1)\mu^2 + \alpha^2} + 1}{\sqrt{\mu^2 + 1}}R\dot{\psi} \approx \frac{(\alpha + 1)(2\alpha - \mu^2)}{\alpha}R\dot{\psi} \tag{21}$$

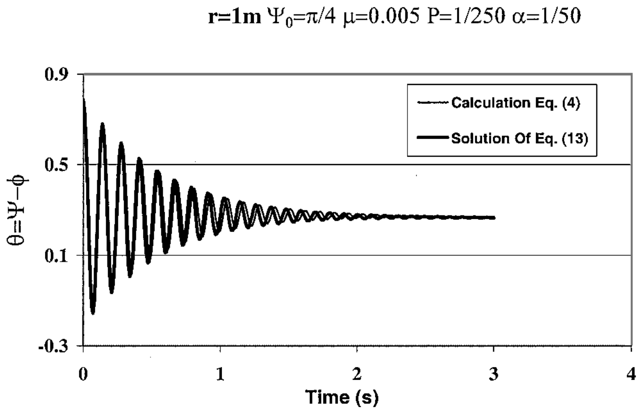

Fig. 5 Comparison of numerical vs analytical results.

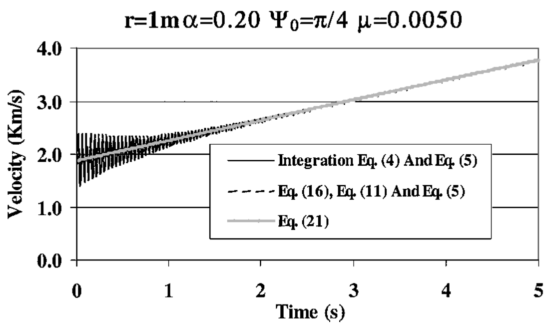

Fig. 6 Comparison of velocities for $\dot{f}/f = \frac{1}{250}$.

and this shows the velocity is asymptotically proportional to $R\dot{\psi}$ for our choice of ψ. Choosing $\dot{f} = 0$, one will find that the asymptotic solution to Eq. (10) will now contain both even and odd powers of q, but the zeroth-order term will still have the solution given in Eq. (11).

Equation (6) and the long time limit found in Eqs. (11) tell us that $b \to 0$, which implies $t_\infty = (b\,\dot{\phi}_0)^{-1} \to \infty$. Thus, the sled will continue to gain velocity for a long period of time whenever the gyration speed varies quadradically with t. An example of this is given in Fig. 6, which shows that both the numerical and the analytical results are in good agreement, which again gives compelling evidence for the decaying factors given in Eq. (17) as well as the value of the phase-lock angle.

So far, we have assumed that the friction coefficient is very small but finite $0 < \mu \ll \alpha < 1$. Equations (17) and (20) show that friction plays a very important role in that it is responsible for damping out the oscillations of the mass sled as it progresses along the circular track. Setting $\mu = 0$ in Eqs. (11) and (21) shows that the long time limits for θ and V become $2n\pi$ and $(\alpha + 1)\,\dot{\psi}R$, respectively.

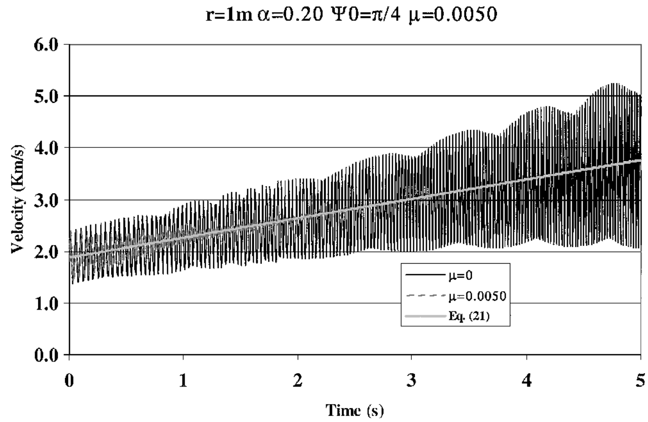

Fig. 7 Comparison of velocities with and without damping $\mu \neq 0$, $\mu = 0$.

Figure 7 displays the influence of friction, where one can see that the velocity no longer damps to the asymptotic limit $V = (\alpha + 1)\dot{\psi}R$ when the force attributed to friction is assumed to be zero.

Conclusions

The analysis presented here gives an explicit expression for the rate at which an accelerating mass becomes phase locked while it traverses a circular slingatron having a gyration phase angle ψ varying as a quadratic in time. All results are based on the assumption that the force attributed to friction is proportional to the normal force $F_{//} = \mu F_\perp$, and the proportionality constant is small $\mu \ll 1$. We have also shown that the phase-lock angle has the simple expression $\theta \to \mu + \mu/\alpha$ for small values of μ and α. To keep θ constant, we must gradually increase $\dot{\psi}$ so that it diverges at $t = (b\dot{\psi}_0)^{-1}$ as does the phase-locked $\dot{\phi}$. This is because as the gyration speed $2\pi R(\dot{f}t + f)$ increases, then so must the mass gain speed $\approx 2\pi\alpha\sin(\theta)$ [see Eq. (5) for $\theta =$ constant] per turn. However, we choose $\psi = \psi_0 + 2\pi f t + \pi\dot{f}t^2$, which causes the gyration speed to become large at large times. Thus, we have shown that the only way the mass can stay phase locked with this increasing gyration speed is to assume smaller values of θ so that it does not gain too much velocity per turn. For our choice of ψ, we have $\theta \to \mu + \mu/\alpha \ll 1$, as time becomes large, thus keeping the accelerating mass phase locked with gyration.

References

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